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Design of movable frame structures using modified Cross procedure

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Research paper

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Design of movable frame structures using modified Cross procedure

An original procedure for static design of movable in-plane frame structures is presented in the paper. The presented design procedure was derived using the modified traditional Cross procedure (TCP). The introduction of the TCP modification has resulted in significant improvement of the design algorithm of movable frame structures as compared to TCP, especially as to elimination of the need to conduct greater number of individual iteration procedures, and to solve linear algebraic equation systems.

Key words:

numerical analysis, relaxation procedure, Cross procedure, division coefficient, transfer coefficient, in-plane frame structure

Prethodno priopćenje

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Proračun pomičnih okvirnih konstrukcija modificiranim Crossovim postupkom

U radu je prikazan izvorni postupak statičkoga proračuna pomičnih ravninskih okvirnih konstrukcija. Prikazani proračunski postupak izveden je modificiranjem klasičnoga Crossova postupka (KCP). Uvedenom modifikacijom KCP-a postignuto je znatno poboljšanje proračunskoga algoritma pomičnih okvirnih konstrukcija u odnosu na KCP, posebice u pogledu uklanjanja potrebe za provedbom većega broja pojedinačnih iteracijskih postupaka i uklanjanja potrebe za rješavanjem sustava linearnih algebarskih jednadžbi.

Ključne riječi:

numerička analiza, relaksacijski postupak, Crossov postupak, razdjelni koeficijent, prijenosni koeficijent, ravninska okvirna konstrukcija

Vorherige Mitteilung

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Berechnung beweglicher Rahmenstrukturen nach modifiziertem Cross-Verfahren

In dieser Arbeit wird das ursprüngliche Verfahren zur statischen Berechnung beweglicher planarer Rahmenstrukturen vorgestellt. Das vorgestellte Berechnungsverfahren wurde durch Modifizieren des klassischen Cross-Verfahrens (KCP) durchgeführt. Die eingeführte Modifikation des KCP erreichte eine signifikante Verbesserung des Berechnungsalgorithmus für bewegliche Rahmenstrukturen in Bezug auf KCP, insbesondere im Hinblick auf die Beseitigung der Notwendigkeit, eine größere Anzahl einzelner iterativer Verfahren durchzuführen, und der Notwendigkeit, Systeme linearer algebraischer Gleichungen zu lösen.

Schlüsselwörter:

numerische Analyse, Relaxationsverfahren, Cross-Verfahren, Verteilungskoeffizient, Übertragungskoeffizient, planare Rahmenstruktur

1. Introduction

In an era when computer facilities were not available worldwide, the procedures for structural design of in-plane static systems were mostly based on the traditional displacement method. Unknowns of this method are values of independent displacements of the system, and, for structures with a large number of nodes, this method leads to a mathematical problem of the linear algebraic equations system, with values of independent general displacements as unknowns.

Using this kind of procedure for the analysis of multi-span and multi-floor in-plane frame structures often led to a problem involving a very large number of equations. In civil engineering practice, solving this mathematical problem was often arduous and lengthy and, therefore, a pressing need was felt to develop alternative methods for solving displacement method equations by eliminating the need to solve a large numbers of linear algebraic equations. Consequently, several procedures have over time been created for finding an iterative solution to equation systems, which involve step by step approximation of the final solution to the problem.

The pioneering approach in this kind of iterative procedures was the procedure proposed by K. Čališev [1-4], where the equation system is solved using the so-called successive approximation technique. In each iteration step of this iterative procedure, Čališev produces an equilibrium state of the considered node by calculating the rotation angle of that node, while the value of bending moment caused by the considered rotation is not calculated directly. Values of moments in the cross section of the structures are derived by using final rotation-angle values of all nodes. The process is repeated for all nodes of the structure so as to establish a satisfactory level of accuracy.

The procedure based on the idea similar to that proposed by Čališev was presented by H. Cross [5-8] in the so-called *moment distribution procedure.* In the Cross procedure, the need for calculating the increment of rotation values for the considered node is eliminated, and only the values of bending moment increments are calculated using the so-called *division coefficients* and *transfer coefficients*.

The procedure based on the Cross procedure, and applied on movable in-plane frame structures, was presented by P. Csonka [8-12]. For some types of structures, the Csonka's procedure exhibits significant acceleration of the Cross procedure. Similar and almost identical procedure for accelerating the Cross procedure was independently created by O. Werner in Zagreb in 1951. (described in [8, 10-12]).

Another iterative procedure for the analysis of movable in-plane frame structures, based on the Ostenfeld's formulation of displacement method, and known as the *single iteration procedure*, was presented by G. Kani in [13].

The procedure presented in this paper is also an iterative procedure for solving equations of the displacement method, which is a kind of extension of the Cross procedure applied on movable in-plane frame structures, and where a significant level of simplification and acceleration is achieved by introducing some changes to and extensions of the original Cross procedure.

2. Existing iteration methods

2.1. Traditional Cross procedure applied on unmovable in-plane frame structures

The traditional Cross procedure (TCP) is the procedure for structural design of in-plane structures, and is a kind of the so-called *relaxation procedures*. According to its mathematical basis, the original variant of this iterative procedure [5-7] is an incremental form of Jacobi's iterative procedure for solving the system of *n* linear algebraic equations with *n* unknowns, where nodal rotation increments are given simultaneously for all nodes of the structure, and where *transferred moment* values obtained in neighbouring cross-sections are used in the next step of iteration. This TCP variant is used in the USA and in many other countries [14-16]. In addition, there is another variant of the Cross procedure in which the increments of nodal rotation are not given simultaneously. In this variant, using the "node by node" technique, transferred moments are immediately used, in the same iteration step, for calculating equilibrium state in other nodes of the structure for which balancing nodal rotation is not as yet defined. This TCP variant constitutes an incremental form of the Gauss-Seidel's iterative procedure, and it is often used in European countries.

The main idea of this kind of procedure is to establish an equilibrium state of the system gradually by rotating the considered node, while at the same time all other nodes in the structure are fixed against rotation in the current design step.

Using this procedure successively, node by node, the equilibrium state is achieved for all nodes of the system in each step of the procedure. The remaining unbalanced moments in each step of the procedure are the moments transferred from the neighbouring nodes of the system and, depending on the selected variant of the procedure, they will be introduced in the current step of iteration, or will be balanced in the next step of iteration.

s-th approximation of the value of bending moments in the end of an arbitrary member i-j can be written as:

$$M_{i,j}^{(s)} = \overline{M}_{i,j} + \sum_{k=1}^{s} \mu_{i,j} \cdot M_i^{(k)} + p \cdot \sum_{k=1}^{s-1} \mu_{j,i} \cdot M_j^{(k)} \quad (1)$$

where: $\overline{M_{i,j}}$ is the fixed-end moment at "i" end of the considered member *i-j* caused by application of an external load, $\mu_{i,j}$ is the so-called Cross division coefficient for "i" end of *i-j* member, $M_i^{(k)}$ is the sum of unbalanced moments in the node "i" of the in-plane structure for the k-th step of iteration, *p* is the so-called Cross transfer coefficient which has a constant value, *p* = 0,5, $\mu_{j,i}$ s the Cross division coefficient for "j" end of *i-j* member, $M_j^{(k)}$ is the sum of unbalanced moments in the node "j" of the frame structure for the k-th step of iteration.

The value of the Cross division coefficient for arbitrary cross section *i-j* of arbitrary node "i" of the considered structure may be expressed as:

$$\mu_{i,j} = -\frac{k_{i,j}}{\sum_j k_{i,j}} \tag{2}$$

where: $k_{i,j} = \frac{EI_{i,j}}{L_{i,j}}$ is the flexural stiffness of the *i-j* member ($EI_{i,j}$ is the product of modulus of elasticity and axial moment of inertia of cross section of the i-j member, $L_{i,j}$ is the length of the *i-j* member), and $\sum_j k_{i,j}$ is is the sum of flexural stiffness values of all members which are connected at the "i" node under study.

The iteration ends when the convergence criteria are established. The convergence occurs when the values of all transferred moments in the s-th step of iteration become lower than some predefined value ε:

$$p \cdot \mu_{j,i} \cdot M_j^{(s)} < \varepsilon \tag{3}$$

for each $\mu_{j,i}\, M_j^{(s)}$ of the system.

For many years, TCP was a powerful design tool for static design of in-plane structures because the need for solving large numbers of linear algebraic equations was eliminated from the design process, and the entire process was reduced to simplest arithmetical operations between division and transfer coefficients, and on the summation of values of individual steps of the procedure.

Today, in modern era, when the procedures for static design of in-plane structures are generally based on matrix formulation of the finite element method, TCP is rarely used as the tool for static design of in-plane structures. However**,** it still remains a very useful tool for the design of simpler frame structures and, consequently, it can be used as the algorithmic basis for creating smaller computer programs for the design of frame structures without using currently available modern commercial computer programs for the static design of structures.

TCP is still an unavoidable part of many study courses of structural design of in-plane structures at faculties of civil engineering throughout the world. For example, A. Kassimali's course (2011) [14] in the USA, R.C. Hibbeler's course (2009) [15], the course of K.M. Leet, C.M. Uang and A.M. Gilbert (2008) [16], are just some of the many study courses in which TCP is an unavoidable theme. In the Republic of Croatia, several study courses of Faculty of Civil Engineering at the University of Zagreb contain, as their unavoidable theme, the TCP, which is presented in a more or less identical form as given in the textbook published by M. Anđelić [8].

2.2. Traditional Cross procedure applied on movable systems

The original algorithm for TCP was created for static design of non-movable in-plane structures without translational displacements (according to displacement method), for example: continuous beams, non-movable frame structures, etc.

Due to the fact that the need for solving equation systems was eliminated and mathematical operations were reduced to just a few simplest arithmetical operations, which are repeated until the satisfactory level of solution accuracy is reached, TCP was used for many years in engineering practice as an easy iterative procedure for structural design of non-movable structures. The fact that TCP can not be used for the design of movable structures, such as in-plane frame structures, represents the main limitation of the initial version of this procedure.

As a consequence, an advanced type of procedure for the design of movable structures has been developed on the basis of the original TCP. Using the original type of TCP, in the first step of this extended type, the restrained structure is designed for external load, as based on the original structure, by introducing restraints to prevent translational displacement of structure, and in which all nodes of the structure are fixed against rotation, and then the bending moments are determined, which are balanced in all nodes. The values of restraining forces for all introduced restraints can be determined using equilibrium equations of forces for isolated parts of structures.

After that, according to the unknown displacements, translational displacements are imposed, one by one, as external load on the restrained structure. Fixed-end moments are calculated, and the extended type of TCP is used for calculating new balanced moments and new restraining forces. At that, as translational displacements are the unknowns, the expressions for the moments and restraining forces will also contain this unknown displacement as an unknown factor. This procedure will be repeated for each unknown translational displacement and, consequently, n+1 individual Cross iteration will be obtained, where "n" is the number of independent translational displacements of the system.

The final value of moments is obtained for each cross section by summing up moments of all individual influences (n+1) of the total restraining force. The number of restraining forces is n, and each final restraining force contains n unknowns- values of translational displacements of the system.

As there are no restraints in the original structure, the value of each final restraining force should be 0. Consequently, the expressions for the restraining force actually become the system of n algebraic equations with n unknowns: n unknown translational displacements of the system. The values of translational displacements, obtained by solving the system of equations, are inserted in expressions for bending moments, and final solutions to the problem are obtained.

The use of such procedure for structures in which the influence of translational displacement is not negligible, i.e. in the design of movable multi-floor frame structures, often led to a lengthy process involving many individual Cross iterations, and to the problem of solving the system of linear algebraic equations with independent translational displacements as unknowns. Because of that, this type of extended TCP was often lengthy

and arduous and, in addition, the main advantage of TCP – no need to solve the system of equations – was eliminated.
The extended TCP applied to the design of movable in-plane frame structures is described in great detail in [8, 14-16].

2.3. Procedure developed by Csonka and Werner

For the static design of movable in-plane frame structures, P. Csonka and O. Werner developed almost identical procedures which resulted in significant acceleration of TCP, and this in such a way that there was no additional need for "n" TCP iterations in in the design of in-plane frame structures with "n" floors. "n" TCP iterations are replaced by the original algorithm in which the influence of translational displacement is replaced by the influence of horizontal forces acting in the line of the beams of the frame structure. These horizontal forces have the same magnitude as the restraining forces for external load, but act in opposite directions.
These procedures are based on the assumption of equality of rotation angle values for all nodes of the same floor and, consequently, half-frame structures are substituted with "n" floors. This half-frame structure is loaded by the said horizontal forces and the flexural stiffness coefficients of its members are derived by summing flexural stiffness coefficients of the corresponding members of the original frame-structure.
The nodes of the half-frame structure are fixed but translational displacements are not restrained.
As shear force values of all columns of the half-frame structure are 0, unknown rotation angles of the nodes are eliminated from equations of the displacement method and, according to TCP, an equilibrium state of moments is achieved by rotating the node under consideration, whereas all other nodes of the structure are fixed against rotation.
Due to the fact that the procedure for calculating influence of horizontal forces (influence of translational displacements of the floors) is applied on the structure with no-restrained translational displacements, the obtained division and transfer coefficients for this procedure differ from the corresponding TCP coefficients.
The transfer coefficient for columns of the structure always amounts to p = -1,0 (except for the hinge support of the first floor column where p = 0), while the coefficient for the beams is p = 0.
The balanced moments, obtained in cross sections of the substituting half-frame structure by using C-W iterative procedure, will be distributed back in cross sections of original frame structure according to the value of the corresponding stiffness coefficient of the obtained cross section member.
Due to wrong assumption on which this procedure is based – that rotation angles for all nodes of the same floor are equal-the moments in the cross sections of original frame structures will be not balanced in the nodes, except in special cases. Consequently, in order to make balanced moments, TCP should be used again for the original frame structure with restrained translational displacements. By this repeated TCP iteration, the new set of restraining forces, which should be equal to 0, will be obtained. However, due to the fact that horizontal forces calculated via TCP are not balanced, the values of these restraining forces will generally not be equal to 0, and obviously there is the need for creating a new set of horizontal forces with the same magnitude as the obtained restraining forces, but with opposite directions. For these horizontal forces, the Csonka – Werner procedure should be conducted once again.
Theoretically, TCP and Csonka – Werner procedures should alternately be applied until a satisfactory state of equilibrium of nodal moments and a satisfactory equilibrium of horizontal forces is obtained.
The number of alternating TCP and Csonka – Werner procedures can be reduced by applying the so-called corrective factor which contains the sum of products of shear forces and floor heights. However, he corrective factor may be used only when conditions of affinity of shear force diagrams for half-frame structure columns have been met.
The final solution for moments of the structure can be obtained by summing moments of first TCP and moments of the second TCP, multiplied by the corrective factor.
A great advantage of the Csonka-Werner procedure is that it is no longer necessary to solve the system of linear algebraic equations nor to calculate translational displacement values. The procedure itself is reduced to simplest arithmetical operations.
The main deficiency of this procedure is the fact that, even in the domain of in-plane frame structures, it can only be used for structures with the same heights of columns at each floor, and with the same column supports at the first floor, but without hinge supports and hinge connections between members of the structure.
A detailed description of this procedure is given in [8, 10-12].

2.4. Kani's procedure

Another procedure for the static design of movable multi-floor frame structures, the so called single iteration procedure, was created by Gaspar Kani [13]. This procedure involves alternating cycles of calculation of balanced nodal moments (by setting, node by node, the corresponding nodal rotations) and cycles of calculation of moments in the columns caused by translational displacements of the beams of the structure.
In Kani's iterative procedure, the equilibrium state is produced on the structure with nodes fixed against rotation, without restraints to prevent translational displacement of structure.
Similar to TCP, Kani introduces the so-called division coefficients for calculating balanced nodal moments, but expressions for these Kani's division coefficients are different from TCP coefficients. In Kani's procedure, the influence of transferred moments is calculated by adding moments from the opposite end of the member, without using the transfer coefficient.

In Kani's procedure, the influence of translational displacements of the beam lines of the structure is calculated using the so-called "translational transfer coefficient", which is derived for each column from the equilibrium condition for horizontal forces of the considered isolated part of structures (created by cutting all columns of the considered floor under the beams). The moments in all columns of the structure can be calculated by multiplying translational transfer coefficients of the considered column with the sum of all moments of the corresponding floor.

One of the main differences between TCP and Kani's procedure is the fact that Kani's procedure produces the so-called "complete iteration" in which the final values of moments, which converge to the desired values, are calculated, while TCP is the so-called "differential iteration" in which the increments of the moments are calculated.

The fact that the final solution can always be produced after just a single iteration is a great advantage of Kani's procedure. The need for calculating restraining forces and for solving the system of algebraic equations is completely eliminated.

Unlike Csonka-Werner procedure, Kani's procedure has not limitation on the height of columns of the first floor of the structure (the columns can vary in height) or on the kind of column supports at the first floor, where the supports can even be a combination of fixed and hinge supports.

In practice, TCP is more comfortable than Kani's procedure because there are alternately cycles of calculating balanced nodal moments using transfer coefficients and cycles of calculating the state of equilibrium of horizontal forces using translational transfer coefficients.

A detailed description of Kani's procedure is given in [17].

3. Modification of Cross procedure

3.1. Fixed-end moments of modified Cross procedure

The modified algorithm is based on the same assumptions on which the original Cross procedure is based: an equilibrium state is achieved on the undeformed shape of the structure, contribution of the translational displacement caused by axial deformation of members is neglected, and contribution of change of cross-sectional shape is also neglected.

Modified procedure described in the paper is derived for the design of in-plane structures that contain horizontal (beams) and vertical (columns) members only, without sloped (inclined) members and, at that, all columns of the first floor have the same height.

The rotation of the member-ends and the corresponding bending moments have "+" sign if they are counterclockwise. The normal force N has a "+" sign if it is a tensile force, and shear force T has a "+" sign if T and N create right-handed Cartesian coordinate system.

The in-plane movable frame structure with five floors and one span (Figure 1), subjected to arbitrary external loads, is considered. The design is carried out by substituting frame structures in which the rotations of all nodes are fixed but, contrary to TCP, without restraints to prevent translational displacements (Figure 1). This substituting frame system is the so-called "basic system". The images of "little squares" (Figure 2) placed in nodes of the basic system represent restraints for preventing nodal rotations only, without preventing translational displacements.

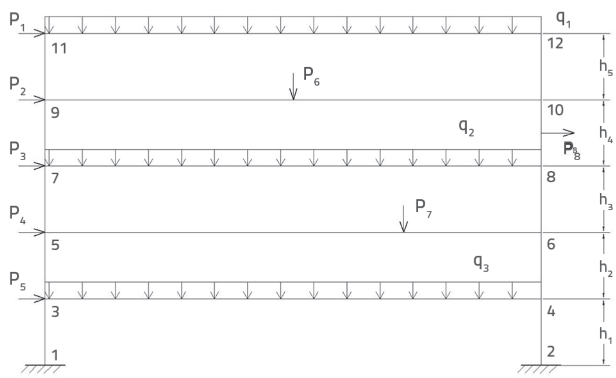

Figure 1. Example of movable multi-floor frame structure

The definition of the "floor" of the frame structure is introduced : the floor is a part of the in-plane frame structure that contains a row of columns and a row of beams. The beams of the considered floor are connected with upper ends of columns. The floors are numbered by numbers : 1, 2, 3 ... , meaning : "floor 1", "floor 2", "floor 3 " ....., where number 1 is the number of the lowest floor.

For example, the third floor contains: columns 7-5, 8-6 and the beam 7-8. The third floor contains nodes 7 and 8.

According to the assumption that axial deformations of the members are neglected, all nodes of a considered floor will obviously have the same magnitude of translational displacement.

The rotation angle of columns ψ_i is defined as follows:

$$\psi_i = \frac{u_{i-1} - u_i}{h_i} \tag{4}$$

where: u_i is the displacement of an arbitrary i-th floor, u_{i-1} is the displacement of an *i*-th floor and h_i is the height of an *i*-th floor (Figure 2), where the angle ψ_i has a "+" sign if it is counterclockwise.

Figure 2 shows the rotation angle of the 3-rd floor columns. Using displacement method equations, the moments of arbitrary i-j column of arbitrary k-th floor, caused by external loads, can be written as: $\psi_3 = \psi_{7\text{-}5} = \psi_{8\text{-}6}$.

$$\tilde{M}_{i,j} = \bar{M}_{i,j} - 6 \cdot k_{i,j} \cdot \psi_k \tag{5}$$

while shear forces can be written as

$$\tilde{T}_{i,j} = \bar{T}_{i,j} - \frac{12 \cdot k_{i,j} \cdot \psi_k}{h_k} \tag{6}$$

where: $\overline{M_{i,j}}$ is TCP's fixed-end moment in *i, j* cross section, k_{ij} is the stiffness coefficient of *i-j* column, h_k is the height of the *k*-th floor, ψ_k is the rotation angle of columns of the k-th floor which contains *i-j* column – caused by translational displacements of the ends of the considered column, $\overline{T}_{i,j}$ is the magnitude of TCP's fixed-end shear force in i,j cross section of *i-j* column (where both ends are fixed against rotation).

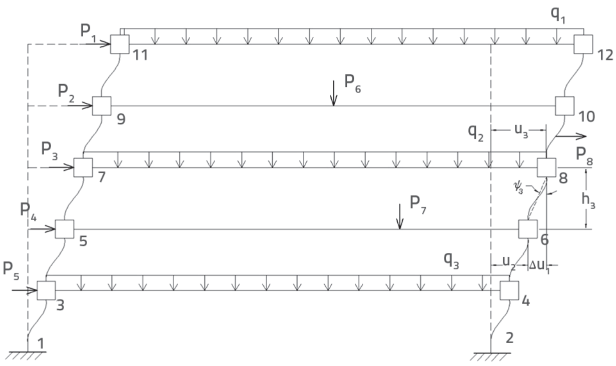

Figure 2. Deformed form of the substituting restrained system

Figure 3. shows an isolated part of the basic system obtained by intersecting 3-rd floor columns at their upper ends.

If we determine by expression (6) the values of transverse forces in sections 7.5 and 8.6 of columns 3 of floor 3, if we introduce the notation H3 for the value of the sum of all horizontal external forces located above the nodes of floor 3 (including forces in these nodes) (in this case H3 = P1 + P2 + P3 + P8), if we include these values in the equation of equilibrium of horizontal actions of this isolated element,

$$\sum F_{i(x)} = H_3 - \tilde{T}_{7,5} - \tilde{T}_{8,6} = 0 \tag{7}$$

where is the sum of all external horizontal forces situated above the 3-rd floor nodes, (including nodal forces) and $\tilde{T}_{7,5}$ and $\tilde{T}_{8,6}$ are shear forces of 3-rd floor columns determined from Eq. (6).

Using Eq. (7), the magnitude of rotation angle of 3-rd flor columns can be written as

$$\psi_3 = \frac{\overline{T}_3 - H_3}{12 \cdot K_3} \cdot h_3 \tag{8}$$

where $\overline{T}_3 = \overline{T}_{7,5} + \overline{T}_{8,6}$ s the sum of TCP's fixed-end shear forces for upper ends of 3-rd floor columns $K_3 = k_{7,5} + k_{8,6}$ is the sum of stiffness coefficients of 3-rd floor columns.

Applying Eq. (5) (combined with eq. (8)) on the cross section 7,5 of column 7-5 of the 3-rd floor, the magnitude of MCP's fixed-end moment caused by external loads for cross section 7,5 of column 7-5 of the 3-rd floor, can be written as:

$$\tilde{M}_{7,5} = \overline{M}_{7,5} + \frac{k_{7,5}}{2 \cdot K_3} \cdot (H_3 - \overline{T}_3) \cdot h_3 \tag{9}$$

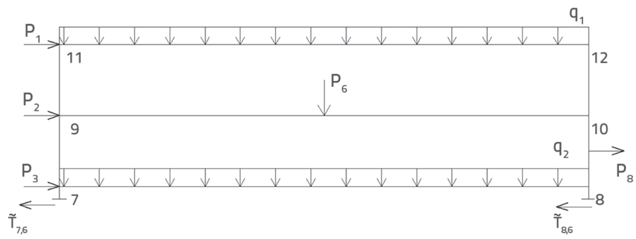

Figure 3. Isolated 4-th and 5-th floors of considered frame structure

The magnitude of MCP's fixed-end moment caused by external load for cross section 7,5 of column 7-5 of the 3-rd floor is given in expression (9). The magnitude of MCP's fixed-end moments for all other cross sections of columns of the 3-rd floor (5,7, 8,6 and 6,8) can be determined by applying Eq. (9) to all other columns of the 3-rd floor.

Based on the above considerations, as the considered 3-rd floor structure has been arbitrarily selected, Eq. (9) can be fully applied to any column at any floor of any in-plane movable frame structure, regardless of the number of columns per floor. For arbitrary in-plane frame structure with arbitrary number of floors, and with arbitrary number of columns, the MCP's fixed-end moment for i,j cross section of arbitrary column i-j of an arbitrarily selected k-th floor, can be written as:

$$\tilde{M}_{i,j} = \overline{M}_{i,j} + \frac{k_{i,j}}{2 \cdot K_k} \cdot (H_k - \overline{T}_k) \cdot h_k \tag{10}$$

where: $\overline{M_{i,j}}$ is the TCP's fixed-end moment for i,j cross section of column i-j of the k-th floor, k_{ij} is the stiffness coefficient of column i-j of the k-th floor, K_k is the sum of stiffness coefficients of all columns of the k-th floor, H_k is the sum of all horizontal external forces acting above the k-th floor (including the forces acting alongside the beams of the k-th floor), $\overline{T}_k$ is the sum of TCP's fixed-end shear forces for upper ends of all columns of the k-th floor, and is the height of the k-th floor.

As nodal translational displacement can not produce beam rotations, the magnitude of the MCP's fixed-end moment for arbitrary u,v cross section of arbitrary beam u-v of arbitrary k-th floor can be written as

$$\tilde{M}_{u,v} = \overline{M}_{u,v} \tag{11}$$

where $\overline{M}_{u,v}$ is the TCP's fixed-end moment for *u,v* cross section.

3.2. Division coefficient of modified Cross procedure

The MCP's fixed-end moments obtained $\tilde{M}_{7,5}$, $\tilde{M}_{7,8}$ i $\tilde{M}_{7,9}$ for cross sections of node seven of the basic system are not balanced and, consequently, they are not the final solution to the original problem because they are calculated for the system with nodes that are fixed against rotation.

An overall unbalanced moment of node seven is the sum of individual moments:

$$M_7^{(0)} = \tilde{M}_{7,5} + \tilde{M}_{7,8} + \tilde{M}_{7,9} \tag{12}$$

where superscript (0) denotes the initial unbalanced state of the moments for node 7, according to step numbering of MCP.
Known expressions for the moments and shear forces of the fixed-end member *i-j* (with length L_{ij}), caused by forced nodal rotation φ_i and φ_j, are:

$$m_{i,j} = 4 \cdot k_{i,j} \cdot \varphi_i + 2 \cdot k_{i,j} \cdot \varphi_j \tag{13}$$

$$t_{i,j} = \frac{6 \cdot k_{i,j}}{L_{i,j}} \cdot \varphi_i + \frac{6 \cdot k_{i,j}}{L_{i,j}} \cdot \varphi_j \tag{14}$$

The moments for cross sections of node 7 of the basic system are produced by defining an angle increment $\Delta\varphi_7^{(1)}$. The sum of these moments has the same magnitude as the unbalanced moment $M_7^{(1)}$, but the sign is opposite (Figure 4). All other nodes of the basic system are fixed against rotation. The superscript "(1)" in $\Delta\varphi_7^{(1)}$ denotes the first step of the iterative procedure.
It is obvious that, for this kind of loading, the sum of all external horizontal forces of arbitrary floor "k" is:

$$H_k(\Delta\varphi_7^{(1)}) = 0 \tag{15}$$

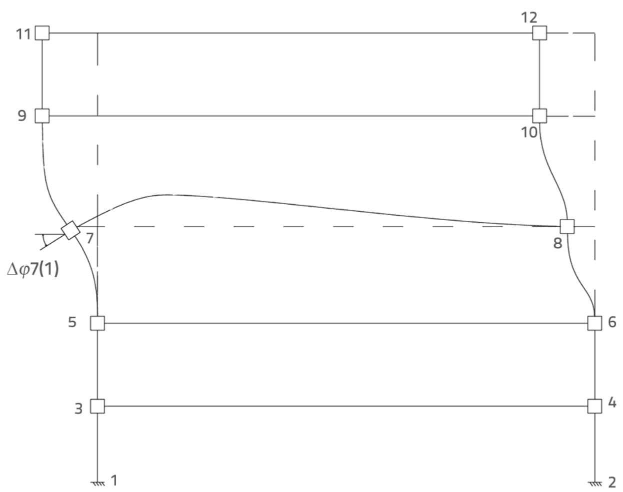

Figure 4. Substituting restrained system subjected to load by rotation of node 7

The increments of the moments that are fixed-end moments caused by $\Delta\varphi_7^{(1)}$ can be calculated by applying (13) to all cross sections of the structure, while all other nodes are fixed against rotation. The increments of the shear forces that are fixed-end shear forces caused by $\Delta\varphi_7^{(1)}$ can be calculated by applying (14) to all cross sections of the structure.
Fixed-end bending moments and shear forces caused by $\Delta\varphi_7^{(1)}$ can be calculated by applying (13) and (14) to the 3-rd and 4-th floors:

$$\begin{aligned} &\bar{M}_{7,5}(\Delta\varphi_7^{(1)}) = 4 \cdot k_{7,5} \cdot \Delta\varphi_7^{(1)}, \bar{M}_{5,7}(\Delta\varphi_7^{(1)}) = 2 \cdot k_{7,5} \cdot \Delta\varphi_7^{(1)}, \\ &\bar{M}_{7,9}(\Delta\varphi_7^{(1)}) = 4 \cdot k_{7,9} \cdot \Delta\varphi_7^{(1)}, \bar{M}_{9,7}(\Delta\varphi_7^{(1)}) = 2 \cdot k_{7,9} \cdot \Delta\varphi_7^{(1)} \\ &\bar{M}_{7,8}(\Delta\varphi_7^{(1)}) = 4 \cdot k_{7,8} \cdot \Delta\varphi_7^{(1)}, \bar{M}_{8,6}(\Delta\varphi_7^{(1)}) = \bar{M}_{6,8}(\Delta\varphi_7^{(1)}) = 0 \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\bar{T}_{7,5}(\Delta\varphi_7^{(1)}) = 6 \cdot \frac{k_{7,5}}{h_3} \cdot \Delta\varphi_7^{(1)}, \bar{T}_{7,9}(\Delta\varphi_7^{(1)}) = 6 \cdot \frac{k_{7,9}}{h_4} \cdot \Delta\varphi_7^{(1)} \\ &\bar{T}_{8,6}(\Delta\varphi_7^{(1)}) = \bar{T}_{8,10}(\Delta\varphi_7^{(1)}) = 0 \end{aligned} \tag{17}$$

where h_3 is the height of floor 3, and h_4 is the height of floor 4.

The sum of the fixed-end shear forces for all columns of the 3-rd floor, and the sum of the fixed-end shear forces for all columns of the 4-th floor, can be written as

$$\begin{aligned} &\bar{T}_3(\Delta\varphi_7^{(1)}) = \bar{T}_{7,5}(\Delta\varphi_7^{(1)}) + \bar{T}_{8,6}(\Delta\varphi_7^{(1)}) = 6 \cdot \frac{k_{7,5}}{h_3} \cdot \Delta\varphi_7^{(1)} \\ &\bar{T}_4(\Delta\varphi_7^{(1)}) = \bar{T}_{7,9}(\Delta\varphi_7^{(1)}) + \bar{T}_{8,10}(\Delta\varphi_7^{(1)}) = 6 \cdot \frac{k_{7,9}}{h_4} \cdot \Delta\varphi_7^{(1)} \end{aligned} \tag{18}$$

By inserting (15), (16) and (17) into general expression (10), the expression for MCP bending moments for cross sections 7,5 and 7,9 of columns 7-5 and 7-9 caused by rotation $\Delta\varphi_7^{(1)}$ of node 7 to which these columns are connected, can be written as:

$$\begin{aligned} &\Delta M_{7,5}(\Delta\varphi_7^{(1)}) = [4 \cdot k_{7,5} - 3 \cdot \frac{(k_{7,5})^2}{K_3}] \cdot \Delta\varphi_7^{(1)} \\ &\Delta M_{7,9}(\Delta\varphi_7^{(1)}) = [4 \cdot k_{7,9} - 3 \cdot \frac{(k_{7,9})^2}{K_4}] \cdot \Delta\varphi_7^{(1)} \end{aligned} \tag{19}$$

where K_3 is the sum of stiffness coefficients for the 3-rd floor and K_4 is the sum of stiffness coefficients for the 4-th floor.

The combination of (11) with (13) yields the expression for the MCP bending moment increment for cross section 7-8 of the beam 7-8 caused by $\Delta\varphi_7^{(1)}$:

$$\Delta M_{7,8}(\Delta\varphi_7^{(1)}) = 4 \cdot k_{7,8} \cdot \Delta\varphi_7^{(1)} \tag{20}$$

Assuming the expression for MCP coefficient $\tilde{a}_{i,j}$ for arbitrary column i-j of an arbitrary k-th floor is defined as:

$$\tilde{a}_{i,j} = 4 \cdot k_{i,j} - 3 \cdot \frac{(k_{i,j})^2}{K_k} \tag{21}$$

where $k_{i,j}$ is the stiffness coefficient of the considered column and K_i is the sum of stiffness coefficients of all columns of the k-th floor to which the column i-j belongs. Now, (19) can be rewritten as follows:

$$\Delta M_{7,5}(\Delta\varphi_7^{(1)})=\tilde{a}_{7,5}\cdot\Delta\varphi_7^{(1)}$$
$$\Delta M_{7,9}(\Delta\varphi_7^{(1)})=\tilde{a}_{7,9}\cdot\Delta\varphi_7^{(1)} \quad (22)$$

Assuming the expression for the MCP coefficient $\tilde{a}_{u,v}$ for an arbitrary beam u-v of an arbitrary k-th floor is defined as:

$$\tilde{a}_{u,v}=a_{u,v}=4\cdot k_{i,j} \quad (23)$$

Now, (20) can be rewritten as follows:

$$\Delta M_{7,8}(\Delta\varphi_7^{(1)})=\tilde{a}_{7,8}\cdot\Delta\varphi_7^{(1)} \quad (24)$$

According to KCP tags, the definition of the MCP "division coefficient" can be introduced for arbitrarily selected cross section i,j of arbitrarily selected node "i",

$$\tilde{\mu}_{i,j}=-\frac{\tilde{a}_{i,j}}{\tilde{A}_i} \quad (25)$$

where $\tilde{a}_{i,j}$ is MCP coefficient for arbitrary cross section i,j of the node "i" that can be calculated for cross sections of columns by applying (21), and for cross sections of beams by applying (23), and $\tilde{A}_i$ is the sum of MCP $\tilde{a}_{i,j}$ coefficients of all members that are connected to node "i". Now, according to KCP, the balancing moment increment $\Delta M_{i,j}^{(s)}(\Delta\varphi_i^{(s)})$ of any s-th MCP iteration step caused by incremental rotation $\Delta\varphi_i^{(s)}$, for any cross section i,j of any node "i", can be written as

$$\Delta M_{i,j}^{(s)}(\Delta\varphi_i^{(s)})=\tilde{\mu}_{i,j}\cdot M_i^{(s)} \quad (26)$$

where $\tilde{\mu}_{i,j}$ is the corresponding MCP division coefficient, and $M_i^{(s)}$ is the sum of unbalanced moments for node "i", for the current s-th MCP iteration step.

According to definitions from KCP and MCP, the moments in Exp. (26) are the so-called MCP distributed moments.

Specifically, using (21), (23), (25) and (26) for cross sections of node 7, the balancing moment increments caused by incremental rotation $\Delta\varphi_7^{(1)}$ .... (distributed moments) for the first iteration step, can be written as

$$\Delta M_{7,5}(\Delta\varphi_7^{(1)})=\tilde{\mu}_{7,5}\cdot M_7^{(1)},\ \Delta M_{7,9}(\Delta\varphi_7^{(1)})=\tilde{\mu}_{7,9}\cdot M_7^{(1)}$$
$$\Delta M_{7,8}(\Delta\varphi_7^{(1)})=\tilde{\mu}_{7,8}\cdot M_7^{(1)} \quad (27)$$

where MCP division coefficients $\tilde{\mu}_{7,5}$ i $\tilde{\mu}_{7,9}$ are calculated by applying (21) and (25), and the coefficientt $\tilde{\mu}_{7,8}$ by applying (23) and (25).

3.3. Transfer coefficients of modified Cross procedure

Except in cross sections of node 7, bending moment increments caused by incremental rotation $\Delta\varphi_7^{(1)}$ also occur in other cross sections of the basic system.

By applying (15) and (16) (for cross sections of the column 8-6) and (18) (for the 3-rd floor) to (10), the bending moment increments for cross sections 6,8 and 8,6 of the column 6-8 caused by incremental rotation $\Delta\varphi_7^{(1)}$can be written as:

$$\Delta M_{6,8}(\Delta\varphi_7^{(1)})=\Delta M_{8,6}(\Delta\varphi_7^{(1)})=-3\cdot\frac{k_{7,5}\cdot k_{6,8}}{K_3}\cdot\Delta\varphi_7^{(1)} \quad (28)$$

where $k_{7,5}$ and $k_{6,8}$ are the stiffness coefficients of columns 7-5 and 6-8 of the 3-rd floor, to which node 7 (rotated by incremental rotation $\Delta\varphi_7^{(1)}$ belongs.

If the corresponding sides of (28) are divided by the corresponding sides of expression (19) for $\Delta M_{7,5}(\Delta\varphi_7^{(1)})$, we obtain:

$$\frac{\Delta M_{6,8}(\Delta\varphi_7^{(1)})}{\Delta M_{7,5}(\Delta\varphi_7^{(1)})}=\frac{\Delta M_{8,6}(\Delta\varphi_7^{(1)})}{\Delta M_{7,5}(\Delta\varphi_7^{(1)})}=\frac{3\cdot k_{6,8}}{3\cdot k_{7,5}-4\cdot K_3} \quad (29)$$

According to the definition from KCP, the MCP "transfer coefficient" $\tilde{p}_{i,j-m,n}$ from the cross section *i, j* of an arbitrary column i-j at an arbitrary k-th floor can be applied to cross sections m,n and *n, m* of the column m-n at the k-th floor, as follows:

$$\tilde{p}_{i,j-m,n}=\tilde{p}_{i,j-n,m}=\tilde{p}_{j,i-m,n}=\tilde{p}_{j,i-n,m}=\frac{3\cdot k_{m,n}}{3\cdot k_{i,j}-4\cdot K_k} \quad (30)$$

where $k_{i,j}$ and $k_{m,n}$ are stiffness coefficients of columns i-j and m-n at the k-th floor, and K_k is the sum of stiffness coefficients for the k-th floor. Then, by applying (46) to (29) for columns 7-5 i 6-8 at the 3-rd floor, we have:

$$\Delta M_{6,8}(\Delta\varphi_7^{(1)})=\Delta M_{8,6}(\Delta\varphi_7^{(1)})=\tilde{p}_{7,5-6,8}\cdot\Delta M_{7,5}(\Delta\varphi_7^{(1)}) \quad (31)$$

where $\tilde{p}_{7,5\text{-}6,8}$ is the MCP transfer coefficient (TCMCP) from cross section 7,5 of column 7-5 to cross sections 6,8, and 8,6 of column 6-8 of the 3-rd floor.

As the 3-rd floor and node 7 of the considered movable frame structure were chosen completely arbitrarily, it follows that the expression can be set for the relationship between incremental bending moments in cross sections of any 2 columns of the considered floor, for any movable frame structure with arbitrary number of floors and arbitrary number of spans.

If the incremental rotation $\Delta\varphi_i$ is set for an arbitrarily selected node "i" of an arbitrary k-th floor of a considered movable frame structure, which causes incremental bending moment $\Delta M_{i,j}(\Delta\varphi_i)$ in the cross section *i, j* of the column *i-j* of *k*-th floor, then, according to expression (31), the corresponding incremental moments for the member ends m-n and n-m of an arbitrarily selected column m-n of the k-th floor can be written as:

$$\Delta M_{m,n}(\Delta\varphi_i) = \Delta M_{n,m}(\Delta\varphi_i) = \tilde{p}_{i,j-m,n}\cdot\Delta M_{i,j}(\Delta\varphi_i) \quad (32)$$

where $\tilde{p}_{ij\text{-}m,n}$ is the MCP transfer coefficient for the transfer from the cross section i,j of the column i-j to cross sections m,n and n,m of the column m-n; here the columns i-j and m-n are the columns of the k-th floor of the structure.

If expression (48) is appropriately applied to cross sections of column 8-10 of the 4-th floor, the incremental bending moments of cross sections 8,10 and 10,8 of the 4-th floor can be written as:

$$\Delta M_{8,10}(\Delta\varphi_7^{(1)}) = \Delta M_{10,8}(\Delta\varphi_7^{(1)}) = \tilde{p}_{7,9-8,10}\cdot\Delta M_{7,9}(\Delta\varphi_7^{(1)}) \quad (33)$$

where $\tilde{p}_{7,9\text{-}8,0}$ is the TCMCP from the cross section 7,9 of column 7-9 to cross sections 8,10 and 10,8 of column 8-1; at that, columns 7-9 and 8-10 are the columns of the 4-th floor.

Due to incremental rotation $\Delta\varphi_7^{(1)}$, incremental moments are also produced at the opposite ends of columns 7-5 and 7-9 (connected to node 7).

If expression (15), expression (16) for cross section 5,7 of column 7-5, and expression (18) for the 3-rd floor, are included in expression (10), then the expression for incremental bending moment of cross section 5,7 of column 7-5, caused by incremental rotation $\Delta\varphi_7^{(1)}$, can be written as:

$$\Delta M_{5,7}(\Delta\varphi_7^{(1)}) = [2\cdot k_{7,5} - 3\cdot\frac{(k_{7,5})^2}{K_3}]\cdot\Delta\varphi_7^{(1)} \quad (34)$$

Division of the corresponding sides of (34) by the corresponding sides of expression (19) for $\Delta M_{7,5}(\Delta\varphi_7^{(1)})$ yields:

$$\frac{\Delta M_{5,7}(\Delta\varphi_7^{(1)})}{\Delta M_{7,5}(\Delta\varphi_7^{(1)})} = \frac{3\cdot k_{7,5} - 2\cdot K_3}{3\cdot k_{7,5} - 4\cdot K_3} \quad (35)$$

If, in accordance with the foregoing, a definition of "transfer coefficient" $\tilde{p}_{ij\text{-}j,i}$ is introduced for transfer from the cross section i,j of an arbitrarily selected column *i-j* of an arbitrary k-th floor to the opposite cross section *j, i* then we have:

$$\tilde{p}_{i,j-j,i} = \tilde{p}_{j,i-i,j} = \frac{3\cdot k_{i,j} - 2\cdot K_k}{3\cdot k_{i,j} - 4\cdot K_k} \quad (36)$$

where k_{ij} is the stiffness coefficient of the column i-j of the k-th floor and K_k is the sum of stiffness coefficients for the k-th floor; then, the application of expression (36) to the column 7-5 of the 3-rd floor and introduction of the result into the expression (35) yields:

$$\Delta M_{5,7}(\Delta\varphi_7^{(1)}) = \tilde{p}_{7,5-5,7}\cdot\Delta M_{7,5}(\Delta\varphi_7^{(1)}) \quad (37)$$

where $\tilde{p}_{7,5\text{-}5,7}$ is the TCMCP from cross section 7,5 of the column 7-5 to the opposite cross section 5,7.

If incremental rotation $\Delta\varphi_i$ is set for an arbitrarily selected node "i" of an arbitrary k-th floor, which would cause incremental moment $\Delta M_{i,j}(\Delta\varphi_i)$ at the cross section i,j of column i-j (connected to the node "i" at its end "i") of the k-th floor , then, according to (37), the corresponding incremental moment (transferred moment) at the opposite end j,i of that column can be written as:

$$\Delta M_{j,i}(\Delta\varphi_i) = \tilde{p}_{i,j-j,i}\cdot\Delta M_{i,j}(\Delta\varphi_i) \quad (38)$$

where $\tilde{p}_{ij\text{-}j,i}$ is the TCMCP from the cross section i,j of an arbitrarily selected column i-j to the opposite cross section j,i of that column.

By appropriate application of expression (38) to the cross section 9,7 of column 7-9 of the 4-th floor, the incremental bending moment at the cross section 9,7 of the 4-th floor can be written as:

$$\Delta M_{9,7}(\Delta\varphi_7^{(1)}) = \tilde{p}_{7,9-9,7}\cdot\Delta M_{7,9}(\Delta\varphi_7^{(1)}) \quad (39)$$

where $\tilde{p}_{7,9\text{-}9,7}$ is the TCMCP from the cross section 7,9 of column 7-9 to the opposite cross section 9,7 of that column.

Finally, due to incremental rotation $\Delta\varphi_7^{(1)}$ the corresponding incremental moment is also produced at the opposite end 8,7 of the beam 7-8 (connected to node 7). Because the incremental rotation $\Delta\varphi_7^{(1)}$ does not produce rotation of the beam 7-8, but only its translational displacement along the horizontal line, the expression for the relationship between increments of bending moments at the ends of the beam 7-8, can be written as:

$$\Delta M_{8,7}(\Delta\varphi_7^{(1)}) = \tilde{p}_{7,8-8,7}\cdot\Delta M_{7,8}(\Delta\kappa_7^{(1)}) \quad (40)$$

where $\tilde{p}_{7,8\text{-}8,7}$ is the TCMCP from one end of the beam 7-8 to the other, which has a constant value of 0.5, just like in TCP.

In general, the value of TCMCP for any beam m-n of any movable frame structure has the same value as the value of the TCP transfer coefficient "p":

$$\tilde{p}_{m,n-n,m} = \tilde{p}_{n,m-m,n} = p = 0,5 \quad (41)$$

and the value of the MCP moment transferred to the opposite end of the beam is the same as the corresponding TCP value:

$$\Delta M_{n,m}(\Delta\varphi_m) = \tilde{p}_{m,n-n,m}\cdot\Delta M_{m,n}(\Delta\varphi_m) = 0,5\cdot\Delta M_{m,n}(\Delta\varphi_m) \quad (42)$$

where $\Delta\varphi_m$ is the increment of the angle of rotation of node "m" to which the beam m-n is connected.

It is obvious that incremental rotation $\Delta\varphi_7^{(1)}$does not produce increments of moments at cross sections of the 1-st, 2-nd and 5-th floors of the considered frame structure (Figure 3). This conclusion results from the application of expression (10) to all columns of the 1-st, 2-nd and 5-th floors, and from the application of expression (11) to all beams of the 1-st, 2-nd and 5-th floors, taking at that into account expression (15) and the fact that for each arbitrarily chosen member i-j of the 1-st, 2-nd and 5-th floors the corresponding values of internal forces caused by $\Delta\varphi_7^{(1)}$ are:

$$\bar{M}_{i,j} = \bar{M}_{j,i} = \bar{T}_{i,j} = \bar{T}_{j,i} = 0 \quad (43)$$

According to definitions from the TCP, the increments of moments in expressions (32), (38) and (42) of the MCP are also named *transferred moments*.

3.4. Steps of modified Cross procedure

Once we have defined MCP parameters (according to TCP definitions): MCP fixed-end moments (expressions (10) and (11)), MCP coefficients $\tilde{a}_{i,j}$ (expressions: (21) for columns and (23) for beams), MCP division coefficients (expression (25)), MCP transfer coefficients (expressions: (30) and (36) for columns and (41) for beams), the MCP can be performed for an arbitrary translationally movable structure with an arbitrary number of floors and spans. The MCP is performed in exactly the same way as the TCP, successively, node by node, where in each step of the procedure all nodes are in an equilibrium state. The remaining unbalanced moments for each step of iteration are the so-called transferred moments from neighbouring nodes of the structure, which will be balanced in the next step of the iteration.

As already shown, the main difference between the MCP and TCP lies in the expressions for: fixed-end moments, coefficients $\tilde{a}_{i,j}$, division and transfer coefficients, where the contribution of translational displacements is built into the MCP.

In TCP, transferred moments are realised only at the opposite ends of the members connected to the node with a given incremental rotation, while in MCP, transferred moments appear at the opposite ends of connected beams and at the ends of all columns of the k-th and (k+1)-th floor (if the k-th floor is not the last floor of the structure), where the considered node with a given incremental rotation belongs to the k-th floor.

The MCP approximation of the bending moment at the end "i" of an arbitrarily selected column i-j of an arbitrary k-th floor can be written as:

$$M_{i,j}^{(s)} = \tilde{M}_{i,j} + \sum_{r=1}^{s} (\tilde{\mu}_{i,j} \cdot M_i^{(r)}) + \sum_{m,n}^{(k)} \sum_{r=1}^{s-1} (p_{m,n-i,j} \cdot \mu_{m,n} \cdot M_m^{(r)}) \quad (44)$$

where: $M_i^{(r)}$ is the overall unbalanced moment for node "i" of the r-th iteration step, $M_m^{(r)}$ is the overall unbalanced moment of the r-th iteration step for an arbitrary node "m" of an arbitrary column m-n of the k-th floor , and $\sum_{m,n}^{(k)} \sum_{r=1}^{s-1} (p_{m,n-i,j} \cdot \mu_{m,n} \cdot M_m^{(r)})$ is the sum of moments transferred from cross sections of all columns of the k-th floor to the cross section i,j (summarized by all iteration steps, with (s-1)-th step as the last step).

In the s-th step, the approximation of the bending moment at the end of an arbitrary beam u-v can be expressed as:

$$M_{u,v}^{(s)} = \tilde{M}_{u,v} + \sum_{k=1}^{s} (\tilde{\mu}_{u,v} \cdot M_u^{(k)}) + 0.5 \cdot \sum_{k=1}^{s-1} (\tilde{\mu}_{v,u} \cdot M_v^{(k)}) \quad (45)$$

where the meaning of individual members of expression (45) is in accordance with the meaning of the corresponding members of expression (1) for TCP.

Just as in TCP, the MCP iteration will end when the convergence criteria according to expression (3) are established.

3.5. Domain of application of MCP

The procedure shown in the paper is suitable for the design of multi-span and multi-floor in-plane frame structures without sloping members of the structure, where there are no vertical translational displacement projections of nodes. In structures with sloping beams or sloping columns, an external load would cause rotation of beams, which would make expressions for the division and transfer coefficients more complicated. However, this is beyond the scope of this paper.

Only the coefficients for the case of vertical columns of the 1-st floor, without members with hinged ends, are shown in this paper.

It is however obvious that MCP can very easily be extended to structures with different heights of columns at the 1-st floor, with hinged ends of any member of the structure, and with any combination of supports of the structure. In sections 2.3 and 2.4 of the paper, known expressions should be used for stiffness coefficients for fixed-end–hinged-end members (derived by applying the static condensation technique) and, for the case of different column heights, use should be made of the fact that all rotation angles of the 1-st floor columns-as displacements of all nodes of the 1-st floor are equal-can be written via one single parameter: for example, via rotation angle of the first column of the 1-st floor.

All these expressions have been derived by the author but, due to space limitations, they are not presented in this text. These expressions will be used in a future publication.

The procedure shown in this paper is suitable for creating a smaller computer program in which all steps of the procedure (that are repetitive in nature) will be performed automatically. Within such a computer program, due to an increased number of MCP transferred moments compared to the number of TCP transferred moments (the moments are transferred to cross sections of all floor columns to which the considered node belongs, and to cross sections of all floor columns above), there is no loss of clarity and efficiency of the procedure in relation to an increase in the number of floors or the number of spans of the structure.

4. Numerical example

4.1. Design using MCP

An example of a two-floor movable in-plane structure for which a moment diagram will be determined using MCP is shown in Figure 5. Cross sections of all members are rectangular, and the width and height (*b/h*) values of individual members are

presented in Figure 5. The modulus of elasticity of the material of all members is assumed equal to $E = 3 \cdot 10^7$ kN/m^2.
Stiffness coefficient values of members of the structure under study are:

$k_{7,8} = k_{4,5} = 15625$ [kNm], $k_{5,6} = 18750$ [kNm], $k_{4,7} = k_{5,8} = 6750$ [kNm]

$k_{4,1} = k_{6,3} = 5062.5$ [kNm], $k_{5,2} = 16000$ [kNm]

Stiffness-coefficient sums for each floor are:

$K_1 = k_{4,1} + k_{5,2} + k_{6,3} = 26125$ [kNm], $K_2 = k_{4,7} + k_{5,8} = 13500$ [kNm]

The values of the TCP fixed-end moments in the cross sections of all members of the structure are obtained by applying well known formulas of the statics of in-plane structures:

$\bar{M}_{7,8} = \bar{M}_{4,5} = 18.00$ [kNm], $\bar{M}_{8,7} = \bar{M}_{5,4} = -18.00$ [kNm]

$\bar{M}_{5,6} = 12.50$ [kNm], $\bar{M}_{6,5} = -12.50$ [kNm]

$\bar{M}_{5,8} = -18.75$ [kNm], $\bar{M}_{8,5} = 18.75$ [kNm]

while these TCP values for all other members that are not subjected to load are equal to 0.
The values of TCP shear forces in cross sections of all columns in the structure are obtained by applying known formulas of the statics of in-plane structures:

$\bar{T}_{5,8} = -25,00$ [kN], $\bar{T}_{8,5} = 25,00$ [kN]

while these TCP values for all other members that are not subjected to load are equal to 0.

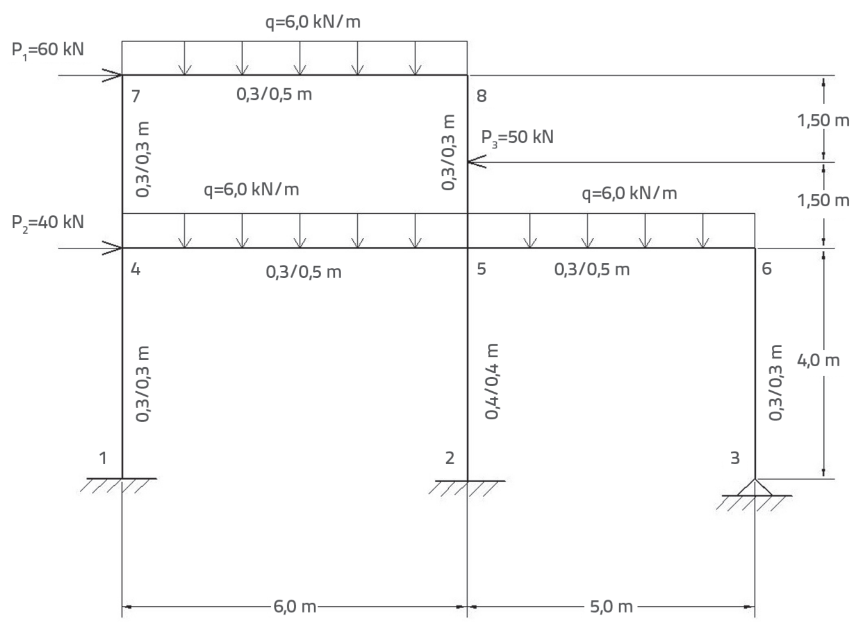

Figure 5. Two-floor frame structure for numerical example

According to the previously presented MCP definitions, the sum of horizontal forces for each floor is:

$H_1 = P_1 + P_2 - P_3 = 50.00$ [kN], $H_2 = P_1 = 60$ [kN]

while the sum of shear forces for each floor is:

$\bar{T}_1 = \bar{T}_{4,1} + \bar{T}_{5,2} + \bar{T}_{6,3} = 0$ [kN], $\bar{T}_2 = \bar{T}_{7,4} + \bar{T}_{8,5} = 25.00$ [kN]

The expression (11) can be used to obtain the values of the MCP fixed-end moments in cross sections of the beams, which are equal to the values obtained for the TCP fixed-end moments of beams.
The expression (10) can be used to obtain values of the MCP fixed-end moments in cross sections of the columns of the structure:

$\tilde{M}_{1,4} = \tilde{M}_{4,1} = \tilde{M}_{3,6} = \tilde{M}_{6,3} = 19.378$ [kNm],

$\tilde{M}_{2,5} = \tilde{M}_{5,2} = 61.244$ [kNm]

$\tilde{M}_{4,7} = \tilde{M}_{7,4} = 26.250$ [kNm], $\tilde{M}_{8,5} = 45.000$ [kNm], $\tilde{M}_{5,8} = 7.500$ [kNm]

MCP division coefficient values for the nodes of the structure can be obtained by appropriate use of expressions (21), (23) and (25):

for node 4: $\tilde{\mu}_{4,7} = -0.1745$, $\tilde{\mu}_{4,1} = -0.1790$, $\tilde{\mu}_{4,5} = -0.6465$

for node 5: $\tilde{\mu}_{5,2} = -0.1831$, $\tilde{\mu}_{5,8} = -0.0893$, $\tilde{\mu}_{5,4} = -0,3307$, $\tilde{\mu}_{5,6} = -0.3969$

for node 6: $\tilde{\mu}_{6,3} = -0.1875$, $\tilde{\mu}_{6,5} = -0.8125$

for node 7: $\tilde{\mu}_{7,8} = -0.7874$, $\tilde{\mu}_{7,4} = -0.2126$

for node 8: $\tilde{\mu}_{8,7} = -0.7874$, $\tilde{\mu}_{8,5} = -0.2126$

No static condensation technique was used for column 6-3, for which node 3 is its hinged support. In the basic system, node 3 is fixed against rotation and, in the iterative procedure, this node is subjected to load by such amount of incremental rotation that it causes disappearance of bending moment in the cross section 3,6. Accordingly, since node 3 has only one cross-section 3,6, the MCP division coefficient for that node is obviously equal to 0: $\tilde{\mu}_{3,6} = -1,0$.
The MCP transfer coefficient values are obtained by using expressions (32) and (36) for the cross sections of the columns:

$\tilde{p}_{4,1-1,4} = \tilde{p}_{6,3-3,6} = 0,4150$, $\tilde{p}_{4,1-5,2} = \tilde{p}_{6,3-5,2} = -0.5374$

$\tilde{p}_{4,1-6,3} = \tilde{p}_{6,3-4,1} = -0.1700$

$\tilde{p}_{5,2-2,5} = 0{,}0752,\ \tilde{p}_{5,2-1,4} = \tilde{p}_{5,2-6,3} = -0.2688$

$\tilde{p}_{4,7-7,4} = \tilde{p}_{8,5-5,8} = 0.2000$

$\tilde{p}_{4,7-5,8} = -0.6000$

$\tilde{p}_{8,5-7,4} = -0.6000$

The previously shown property of transfer coefficients is used:

$\tilde{p}_{i,j-j,i} = \tilde{p}_{j,i-i,j}$ from (36) and $\tilde{p}_{i,j-m,n} = \tilde{p}_{i,j-n,m}$ i $\tilde{p}_{i,j-m,n} = \tilde{p}_{j,i-m.n}$ from (30).

According to expression (57), the values of the MCP transfer coefficients for cross sections of all beams of the frame structure are equal to the corresponding TCP transfer coefficients, and amount to: 0.5000.

The calculation scheme of the MCP iterative procedure is shown in Figure 6. The calculation scheme is created in accordance with the usual form of TCP calculation schemes as used in engineering practice in the Republic of Croatia, and also in accordance with the schemes depicted in [8]. The iteration was performed according to the Gauss-Seidel procedure, in which the transferred moments are immediately introduced into calculation in the ongoing iteration step. A "square" (with the values of the MCP division coefficients of that node) is drawn for each node of the structure, and the values of the initial MCP fixed-end moments are entered below cross sections of each member of the structure. In the scheme shown in Figure 6, the MCP transfer coefficients are also entered accordingly.

The results of the MCP iterative procedure presented in the paper are rounded to four significant digits.

The MCP iterative procedure begins by balancing the moments in node 5. The initial unbalanced moment of node 5 is the sum of MCP fixed-end moments in the cross sections of node 5, whose magnitude is: +63.2441 kNm. By multiplying this unbalanced moment by division coefficients for node 5, the balancing increments of moments are obtained: for a cross section 5,2: -11.5800 kNm, for a cross section 5,4: -20.9148 kNm, for a cross section 5,8 -5.6477 kNm, for a cross section 5,6: -25.1015 kNm. By subsequently multiplying these distributed increments of node 5 moments by the corresponding MCP transfer coefficients, transferred increments of moments are obtained as follows: for cross section 2,5: -0.8708 kNm, for cross sections 4,1 and 1,4: +3.1127 kNm, for cross sections 6,3 and 3,6: +3.1127 kNm, for cross section 4,5: -10.4574 kNm, for cross section 6,5: -12.5508 kNm, for cross section 8,5: -1.1294, kNm, and for cross sections 4,7 and 7,4: +3.3866 kNm.

The unbalanced moment of node 8 is the sum of MCP fixed-end moments in cross sections of that node and transferred moments from node 5, and its value is: +25.8704 kNm. If this unbalanced moment is multiplied by division coefficients for node 8, balancing increments of moments are obtained as follows: for cross section 8,5: -5.5000 kNm, and for cross section 8,7: -20.3704 kNm. By subsequently multiplying these distributed increments of the node 8 moments by the corresponding MCP transfer coefficients, transferred increments of moments are obtained as follows: for cross section 7,8: -10.1852 kNm, for cross section 5,8: -1.1000 kNm, and for cross sections 4,7 and 7,4: -3.3000 kNm.

The unbalanced moment of node 7 is also calculated. This moment is equal to the sum of MCP fixed-end moments in cross sections of that node, and previously transferred moments from nodes 5 and 8, and it amounts to: 40.7534 kNm. By multiplying this unbalanced moment by division coefficients for node 7, balancing increments of moments are obtained as follows: for cross section 7,8: -32.0893 kNm, and for cross section 7,4: -8.6641 kNm. By subsequently multiplying these distributed increments of node 7 moments by the corresponding MCP transfer coefficients, the following transferred increments of moments are obtained: for cross section 8,7: -16.0445 kNm, for cross section 4,7: -1.7328 kNm, for cross sections. 5,8 and 8,5: +5.1984 kNm.

The unbalanced moment of node 4 is also calculated. It is equal to the sum of MCP fixed-end moments in cross sections of that node and previously transferred moments from nodes 5, 7 and 8, and it amounts to: +61.2376 kNm. By multiplying this unbalanced moment

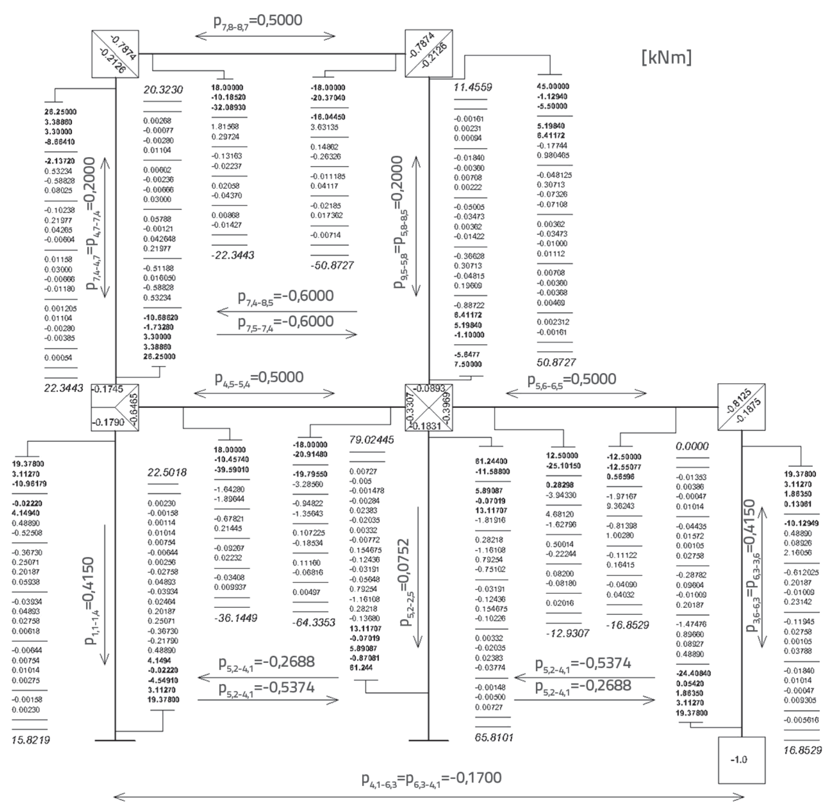

Figure 6. Calculation scheme of MCP iterative procedure for numerical example

by division coefficients for node 4, the following balancing increments of moments are obtained: for cross section 4,5: -39.5901 kNm, for cross section 4,1: -10.9618 kNm, and for cross section 4,7: -10.6862 kNm. By subsequently multiplying these distributed increments of node 4 moments by the corresponding MCP transfer coefficients, the following transferred increments of moments are obtained: for cross section 5,4: -19.7955 kNm, for cross section 1,4: -4.5491 kNm, for cross sections 5,2 and 2,5: +5.8909 kNm, for cross sections 3,6 and 6,3: +1.8635 kNm, for cross section 7,4: -2.1372 kNm, and for cross sections 5,8 and 8,5: +6.41172 kNm.

The unbalanced moment of node 6, equal to the sum of MCP fixed-end moments in cross sections of that node and previously transferred moments from nodes 4 and 5, is also calculated. By multiplying this unbalanced moment amounting to: -0.6969 kNm by division coefficients for node 4, the following balancing increments of moments are obtained: for cross section 6,5: +0.56596 kNm, and for cross section 6,3: +0.13061 kNm. By subsequently multiplying these distributed increments of node 6 moments by the corresponding MCP transfer coefficients, the following transferred increments of moments are obtained: for cross section 5,6: +0.28298 kNm, for cross section 3,6: +0.0542 kNm, for cross sections 1,4 and 4,1: -0.0222 kNm, and for cross sections 2,5 and 5,2: -0.070188 kNm.

The moment of the supporting node 3, equal to the sum of MCP fixed-end moments in the cross section of that node and previously transferred moments from nodes 4, 5 and 6, is also calculated. Multiplying this supporting moment amounting to +24.4084 kNm by the division coefficient for node 3, amounting to -1.0, the increment of supporting moment for the cross section 3,6 is obtained: -24.4084 kNm. Thus the total supporting moment at that hinged support 3 assumes the value of 0. By subsequently multiplying this distributed increment of node 3 moment by the corresponding MCP transfer coefficients, the following transferred increments of moments are obtained: for cross section 6,3: -10.1295 kNm, for cross sections 1,4 and 4,1: +4.1494 kNm, and for cross sections 2,5 and 5,2: +13.11707 kNm.

This last action completes the first round of the iterative procedure. All values of distributed and transferred moments of the first iterative round are highlighted in bold letter type in Figure 5. The second iteration round begins in the same way as the first one, by balancing the remaining unbalanced moments of node 5 from the first iterative round. The total unbalanced moment for node 5 for the second iterative round is the sum of all transferred moments from nodes 3,4,6,7 and 8, from the first round. By multiplying this unbalanced moment by division coefficients for node 5, balancing increments of moments for the second iterative round are obtained. Unbalanced moments in other nodes of the structure will be balanced in the same order as in the first iterative round. Unbalanced moments are the sum of the moments transferred from the previous iterative round. The procedure in the second iterative round, including the node balancing order, is completely identical to the first iterative round. The same applies to all other subsequent iterative rounds.

The procedure ends with the iterative round for which the values of all transferred moments are less than some predefined value.

In the numerical example shown, this predefined value is 0.1000 kNm, and so the procedure ends with the 5-th round.

Final moments for each cross section of the structure are obtained as the sum of all distributed and all transferred moments of all iterative rounds, without unbalanced transferred moments of the last iterative round, which are neglected. Therefore, the final values of the moments are obtained by simply summing all the values of the column that corresponds to the observed cross section in the calculation scheme (Figure 6).

The final values of the moments, rounded to the fourth decimal place, shown in the diagram in Figure 7, are obtained by summing values obtained from the calculation scheme (Figure 6).

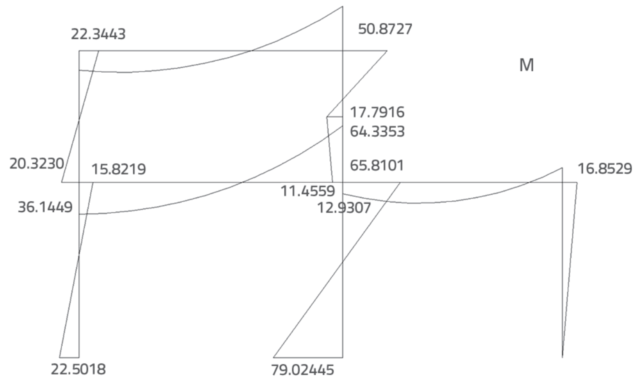

Figure 7. M diagram for numerical example solved according to MCP

4.2. Analysis by TCP for translationally movable structures

To enable comparison with MCP, the solution of the same numerical example was performed by TCP for translationally movable structures.

The TCP will be conducted using steps described in [8], Section 7.1.2. However, some corresponding calculation parameters are marked differently. The results are rounded to the fourth decimal place, just like in MCP.

A restrained system in which all nodal rotations and all independent translational displacements are prevented is created (Figure 8): displacement of beam line of the 1-st floor $\mathbf{u}_1$ and displacement of beam line of the 2-nd floor $\mathbf{u}_2$.

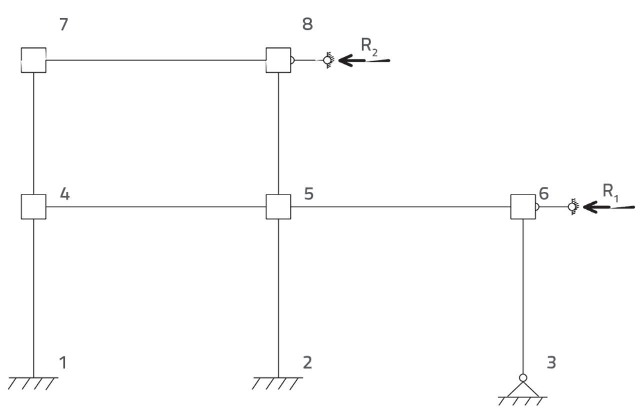

Figure 8. Restrained system for numerical example according to TCP

Using known expressions, fixed-end moments are obtained at all cross sections of the restrained system: $\bar{M}_{i,j}$. For all cross sections of the restrained structure, these moments are identical to the moments given in Section 3.1.

Using the already obtained stiffness coefficient values of members of the structure, as presented in [8], the values of TCP division coefficients are obtained:

for node 4: $\mu_{4,7}$ = -0.2460, $\mu_{4,1}$ = -0,1845, $\mu_{4,5}$ = -0.5695

for node 5: $\mu_{5,2}$ = -0.2801, $\mu_{5,8}$ = -0.1182, $\mu_{5,4}$ = -0.2735, $\mu_{5,6}$ = -0.3282

for node 6: $\mu_{6,3}$ = -0.1684, $\mu_{6,5}$ = -0.8316

for node 7: $\mu_{7,8}$ = -0,6983, $\mu_{7,4}$ = -0.3017

for node 8: $\mu_{8,4}$ = -0.6983, $\mu_{8,5}$ = -0.3017

In doing so, the expressions derived using static condensation technique are used for member 3-6, provided that the supporting moment in node 3 is equal to 0.

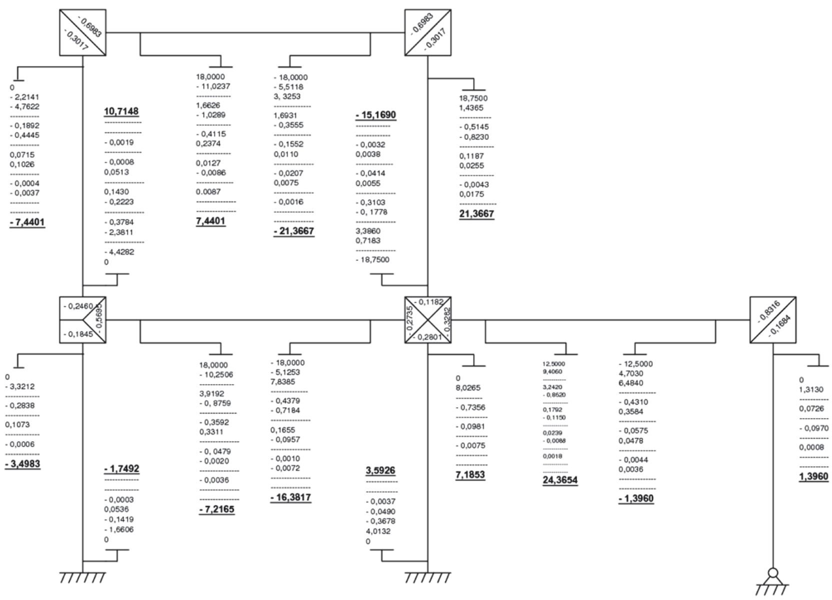

Figure 9. Schematic view of TCP iterative procedure for numerical example according to TCP, for external load

Using TCP (Figure 9), the moments in cross sections of the restrained structure, caused by external load, are obtained: $M_{i,j}(0)$. The values of these moments are indicated in Figure 9 using bold letter type.

A known expression is used for the shear force value at the ends of the member:

$$T_{i,j} = \frac{M_{i,j} + M_{j,i}}{h_{i,j}} + T_{i,j}^{(0)}$$

where $T_{i,j}^{(0)}$ is the value of shear force for a simply supported beam caused by external load, and $h_{i,j}$ is the length of the member i – j (in this case the height of the column i – j); shear force values $T_{i,j}(0)$ in the cross sections of the columns of the restrained system are obtained from values of the moments $M_{i,j}(0)$.

Beam lines of individual floors are isolated by cutting all columns of a particular floor and the floor above it, and by releasing shear forces of the columns. Restraining force values of the restrained system are obtained by applying equilibrium equations for the horizontal forces acting on the isolated beam line (Figure 8): for the 1-st floor: $R_1(0) = P_2 - T_{4,1}(0) - T_{5,2}(0) - T_{6,3}(0) + T_{4,7}(0) + T_{58}(0)$ = 10.4488 kN, and for the 2-nd floor: $R_2(0) = P_1 - T_{7,4}(0) - T_{8,5}(0)$ = 37.8196 kN.

To determine values of previously prevented displacements, joints are placed in all nodes of the restrained system, and appropriate displacement schemes are created (Figure 10). The translational displacement $\mathbf{u_1}$ is set (Figure 10.a).

Values of fixed-end moments $M_{ij}(u_1)$, caused by displacement u1By, are obtained using kn $M_{ij}(u_1)$ own expressions for fixed-end moments at the ends of the member, caused by rotation of the member, and, in addition, using static condensation for member 3-6 (Figure 11).

With these values as initial values, the TCP is performed once again (Figure 11). IAt the end of iterative TCP process (Figure 11), the moments caused by displacement $\mathbf{u_1}$: $M_{i,j}(u_1)$ are obtained in cross sections of the restrained system, as shown in Figure 11.

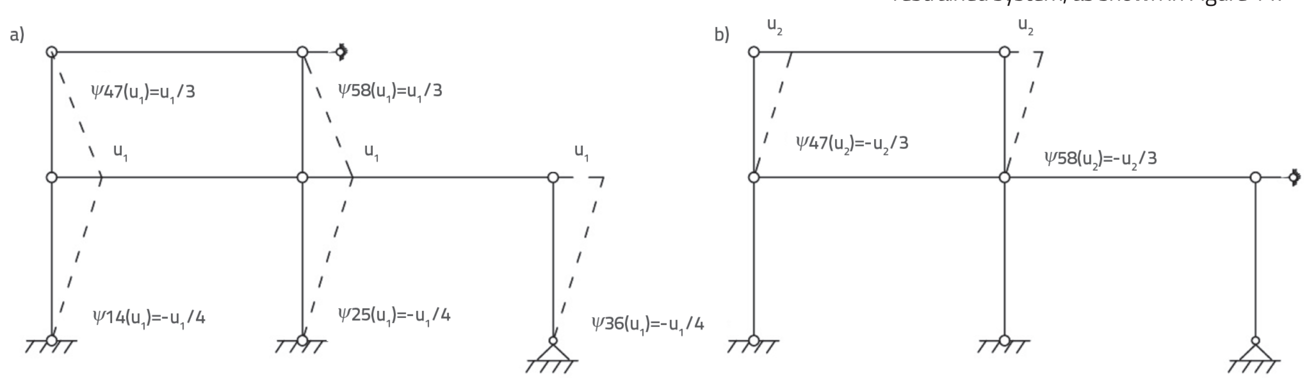

Figure 10. Displacement scheme for numerical example, for displacement u_1 and u_2

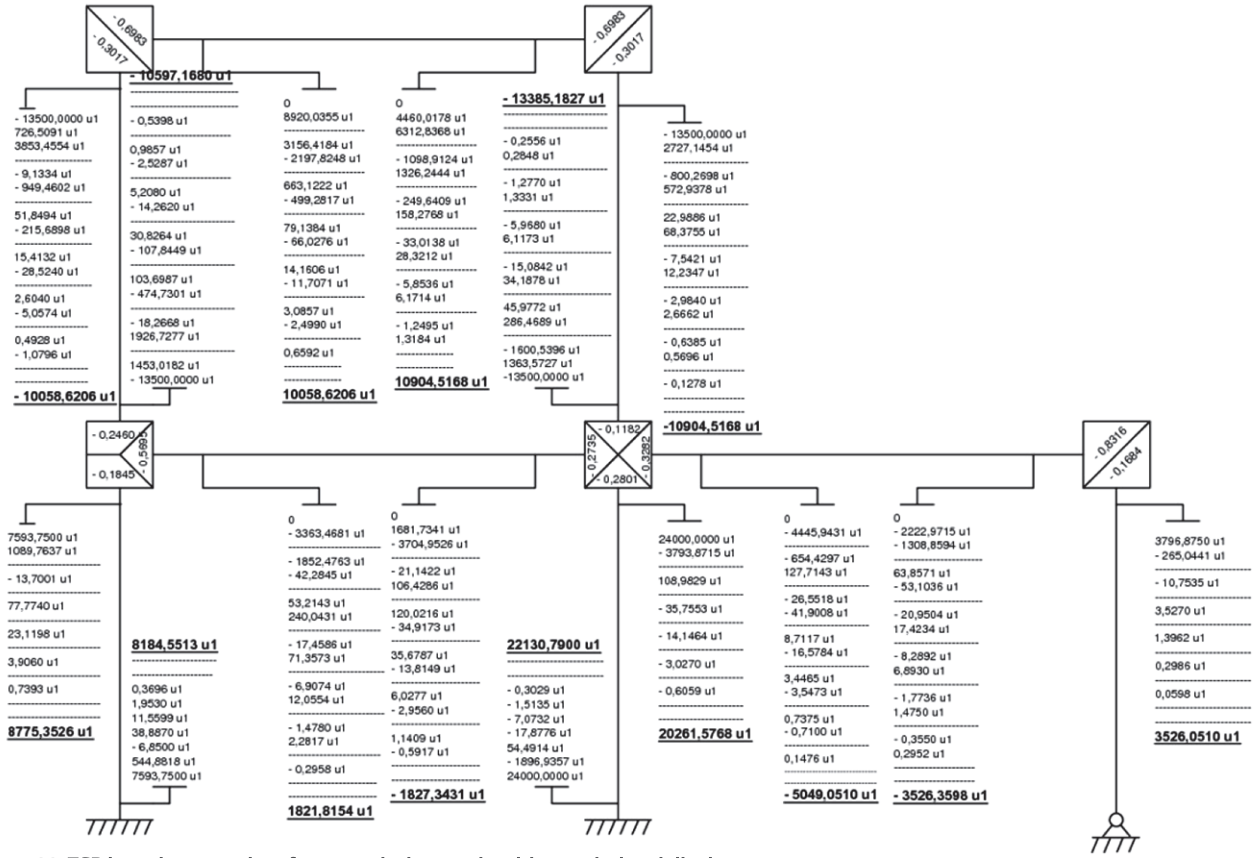

Figure 11. TCP iterative procedure for numerical example with translational displacement u_1

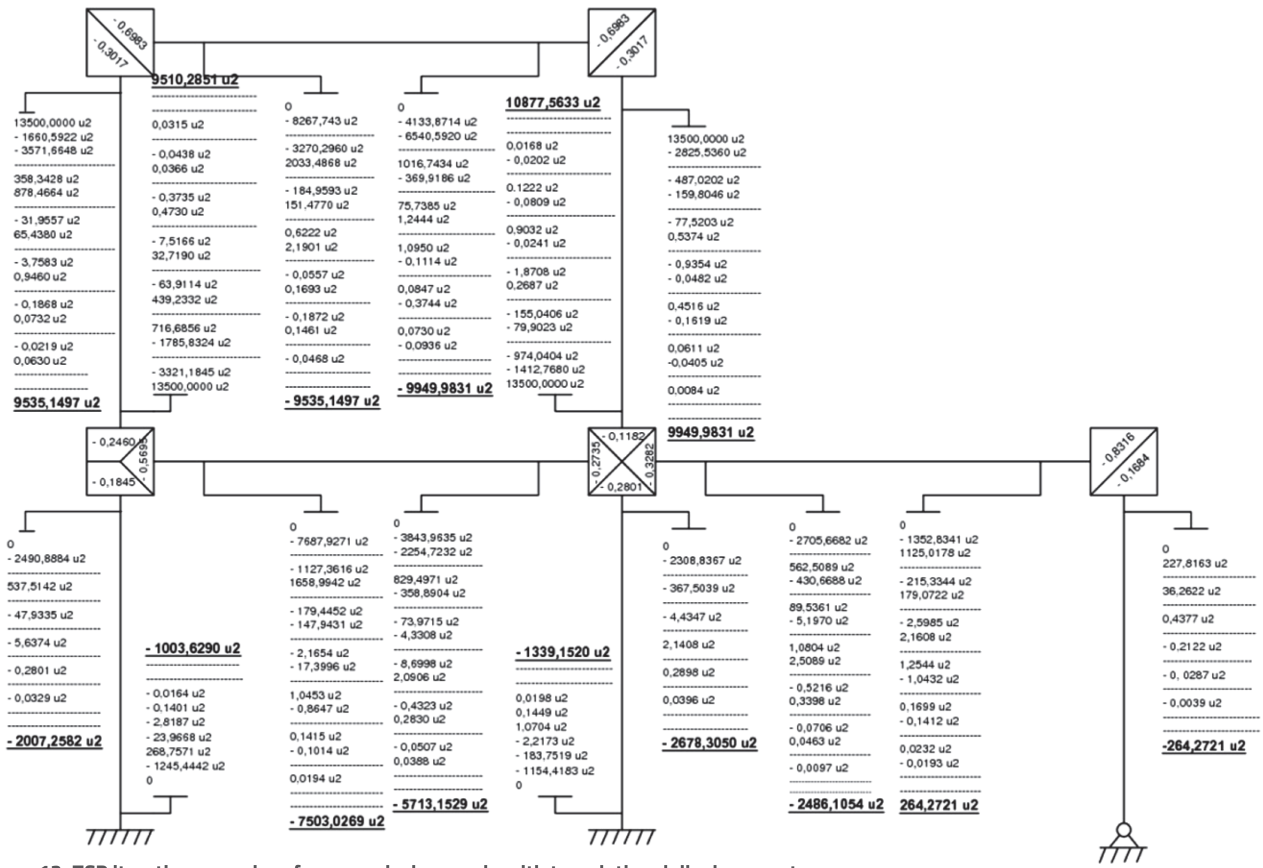

Figure 12. TCP iterative procedure for numerical example with translational displacement u_2

In accordance with the case already shown for external load, the values of shear forces $T_{i,j}(u_1)$ are obtained in cross sections of the columns from the values of $M_{i,j}(u_1)$.

Just like in the case shown for external load, the values of restraining forces of the restrained system are obtained by applying equilibrium equations for horizontal forces acting on the isolated beam lines (the difference being that there is no contribution of forces P_1 and P_2 in the equilibrium equations): for the 1-st floor: $R_1(u_1) = -30701.4866 \cdot u_1$, for the 2-nd floor: $R_2(u_1) = 14981.8293 \cdot u_1$.

The translational displacement $\mathbf{u_2}$ is also set (Figure 10 b)).

The values of moments $m_{i,j}(u_2)$ caused by the displacement $\mathbf{u_2}$ are shown in Figure 12.

The TCP is performed again with these values as initial values (Figure 12). At the end of the iterative TCP (Figure 12), the moments are obtained in cross sections of the restrained system caused by displacement $\mathbf{u_2}$: $M_{i,j}(u_2)$. The values of these moments are marked with bold letter type in Figure 12.

Just as in the case shown for displacement $\mathbf{u_1}$, the values of restraining forces of the restrained system are obtained by applying equilibrium equations for horizontal forces acting on isolated beam lines: for the 1-st floor: $R_1(u_2) = 14982.0119 \cdot u_2$, and for the 2-nd floor: $R_2(u_2) = -13290.9937 \cdot u_2$.

The final values of restraining forces are obtained by summing all the contributions:

$R_i = R_i(0) + R_i(u_1) + R_i(u_2)$, where i = 1,2 is the numbering for floors. According to the fact that there are no restraints in the original structure, the final value of each restraining force should be equal to 0: $R_i = 0$. Using this condition, the following equations are obtained:

$$R_1 = 10.4488 - 30701.4866 \cdot u_{1\,+}\, 14982.0119 \cdot u_2 = 0$$
$$R_2 = 37.8196 + 14981.8293 \cdot u_1 - 13290.9937 \cdot u_2. = 0$$

The obtained equations represent a system of two linear algebraic equations with two unknows, where the unknows are the required values of translational displacements of beam lines of the structure.

By solving this system of equations, the following values of required displacements are obtained: $u_1 = 384.2629 \cdot 10^{-5}$ m i $u_2 = 717.6981 \cdot 10^{-5}$ m.

The final values of moments in cross sections of the structure are obtained by summing all contributions:

$M_{i,j} = M_{i,j}(0) + M_{i,j}(u_1) + M_{i,j}(u_2)$. By including the obtained values of translational displacements in this expression, the final moments in all cross sections of the structure are obtained. These results are shown in Table 1 along with the results of other procedures.

4.3. Comparison of procedures

Table 1. shows the values of moments in cross sections of the structure from numerical example shown in Figure 5 for TCP and MCP.

Table 1. Comparative results of numerical example for MCP and TCP

Procedure / Cross section	TCP	MCP
$M_{1.4}$	22.4980	22.5018
$M_{2.5}$	79.0219	79.0244
$M_{4.1}$	15.8159	15.8219
$M_{4.7}$	20.3177	20.3230
$M_{4.5}$	-36.1336	-36.1449
$M_{5.2}$	65.8209	65.8101
$M_{5.8}$	11.4648	11.4559
$M_{5.4}$	-64.4067	-64.3353
$M_{5.6}$	-12.8790	-12.9307
$M_{6.3}$	16.8432	16.8529
$M_{6.5}$	-16.8432	-16.8529
$M_{7.4}$	22.3419	22.3443
$M_{7.8}$	-22.3419	-22.3443
$M_{8.5}$	50.8755	50.8727
$M_{8.7}$	-50.8755	-50.8727

The comparison of results shows that the largest relative deviation of MCP results from TCP results is: 0.4014 %, and the smallest relative deviation is: 0.0032 %, which is explained by the accumulation of errors in the process of rounding of individual results.

A comparison of MCP and TCP reveals significant advantages of MCP over TCP: 1. Regardless of the number of floors and number of spans of the frame structure, MCP reaches final solutions after only one iterative procedure, while in TCP it is necessary to perform (n+1) individual iterative procedures (where "n" is the number of floors of the structure). This fact makes the MCP evidently more time-efficient, more transparent, and more effective procedure; 2. Unlike TCP, MCP does not need to calculate shear forces in columns, nor to calculate restraining forces, thus further reducing the time required for calculation, and increasing efficiency compared to TCP; 3. The biggest advantage of MCP over TCP is the complete elimination of linear algebraic equations to find the values of unknown translational displacements, where the procedure, from start to end, remains in the domain of simplest mathematical procedures, reduced to the simplest arithmetic operations with numbers.

If we analyse the number of steps required, i.e. the number of individual sub-procedures within TCP, then it follows that TCP contains the following sub-procedures: calculation of values of the fixed-end moments and values of division and transfer coefficients; iterative procedure performed (n+1) times; calculation of shear forces values of columns, performed (n+1) times; calculation of restraining values, performed (n+1) times; solving a system of n linear algebraic equations with n unknowns; using obtained values of translational displacements, and summation of all individual contributions to obtain the final solution.

Therefore, for an n-floor structure, it is necessary to perform 3 + 3 (n + 1) single steps (sub-procedures), if the design is carried out using the TCP.

On the other hand, the MCP always contains only two individual steps (sub-procedures): calculation of values of fixed-end moments and values of division and transfer coefficients; an iterative procedure which, regardless of the number of floors, is always carried out only once. The advantage of MCP over TCP in terms of the required number of steps (number of individual sub-procedures) is more than obvious. For example, for the frame structure with n = 5 floors: 3 + 3x(5 + 1) = 21 individual steps (sub-procedures) are required to implement TCP, while MCP (always) requires only two steps.

5. Conclusion

As shown in the paper, MCP has significantly improved TCP in the design of in-plane movable structures. The basic idea of TCP to achieve equilibrium state by successive calculation of *distributed* and *transferred* moments within an iterative procedure, by gradually approaching the correct solution, is retained in MCP as well, with the exception that, unlike TCP, the equilibrium state is achieved on a basic system that prevents nodal rotations, but horizontal displacements of the nodes are not prevented (Figure 1.b). This significant difference in the choice of the basic system results in completely different expressions for the division and transfer coefficients of MCP compared to TCP.

In the modified procedure, the final solution is obtained after only one iterative process, unlike traditional procedure, in which the final solution is obtained by summing the results of several individual iterative processes. The number of required individual iterative processes in TCP is (n+1), where "n" is the number of floors of the observed movable frame structure, so that an enormous saving of calculation time is obvious. In addition, unlike TCP, the need to solve a system of "n" linear equations with "n" unknowns (where "n" is the number of floors of the frame structure) is eliminated in MCP, thus further simplifying the procedure.

In the light of the above, it be concluded that MCP is suitable as an algorithmic basis for creating smaller computer programs for the design of translationally movable frame structures, regardless of the number of spans and the number of floors of the frame, without the use of commercial computer programs.

Compared to the Csonka and Werner procedure, MCP is immeasurably more dominant for a wider range of frame structures because, unlike the Csonka and Werner procedure, MCP is not limited to frame structures with equal column heights of the first floor. There is no restriction on the necessary identity of the supports of the first floor columns. Also, MCP has no restrictions on possible existence of joints between any members of the frame structure. Precisely, in relation to this limitation of Csonka and Werner procedure, it was not possible to compare in this paper the presented MCP with the procedure of Csonka and Werner, because the numerical example contains a combination of fixed and hinged supports of the first floor columns which, as shown, can not be solved by the procedure of Csonka and Werner.

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